Journal of Global Optimization (2005) 32: 451–466
DOI 10.1007/s10898-003-2683-2

A Nonlinear Scalarization Function and Generalized Quasi-vector Equilibrium Problems

This paper is dedicated to Professor Franco Giannessi for his 68th birthday

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(Received 15 August 2003; accepted 19 August 2003)

Abstract. Scalarization method is an important tool in the study of vector optimization as corresponding solutions of vector optimization problems can be found by solving scalar optimization problems. In this paper we introduce a nonlinear scalarization function for a variable domination structure. Several important properties, such as subadditiveness and continuity, of this nonlinear scalarization function are established. This nonlinear scalarization function is applied to study the existence of solutions for generalized quasi-vector equilibrium problems.

Mathematics Subject Classifications. 90C47, 90C33

Key words: existence, generalized quasi-vector equilibrium, scalarization method

1. Introduction and Preliminaries

The equilibrium problem is a generalization of classical variational inequalities. This problem contains many important problems as special cases, including optimization, Nash equilibrium, complementarity, and fixed point problems, (see [4, 5, 17] and the references cited therein). Recently, there has been an increasing interest in the study of vector equilibrium problems. Many results on existence of solutions for vector variational inequalities and vector equilibrium problems have been established (see [3, 6, 7, 11–14, 16, 17, 20]). These results are derived under assumptions of some kind of monotonicity and pesudomonotonicity. In these studies, the ordering cone is assumed to be a fixed, closed and convex cone.

Solution concept with variable domination structures was introduced by Yu [22]. It is a generalization of the solution concept with the fixed domination structure in multicriteria decision making problems. On the other hand, another type of the solution concept with variable domination structures was introduced in [6]. Recently, in [8], a nonlinear scalarization function with two variables was explored.

In this paper, we introduce a scalarization function of vector-valued mappings under a variable ordering (domination) structure. This new nonlinear scalarization function includes the ones in [8, 10] as special cases. By using this scalarization function, we establish existence of solutions for a class of generalized vector quasi-equilibrium problems without any conditions of monotonicity type. It is worth noting that this existence result includes the classical result obtained in [9] as a special case.

Let E and X be locally convex Hausdorff vector topological spaces, and $Y \subset E$ a nonempty subset. Let $F: Y \to 2^X$ be a set-valued map.

DEFINITION 1.1 [1].

(i) F is called upper semi-continuous at $y_0 \in Y$ if, for any neighborhood $N(F(y_0))$ of $F(y_0)$, there exists a neighborhood $N(y_0)$ of y_0 such that

$$\forall y \in N(y_0), \quad F(y) \subset N(F(y_0));$$

(ii) F is called upper semi-continuous on Y if F is upper semi-continuous at every $y \in Y$;

(iii) F is called lower semi-continuous at $y_0 \in Y$ if, for any $x_0 \in F(y_0)$ and any neighborhood $N(x_0)$ of x_0, there exists a neighborhood $N(y_0)$ of y_0 such that

$$\forall y \in N(y_0), \quad F(y) \cap N(x_0) \neq \varnothing;$$

(iv) F is called lower semi-continuous on Y if F is lower semi-continuous at every $y \in Y$;

(v) F is called continuous at $y_0 \in Y$ (respectively, on Y) if F is both upper semi-continuous and lower semi-continuous at y_0 (respectively, on Y).

DEFINITION 1.2. The set-valued map F is closed if it graph,

$$\mathrm{Graph} F = \{(y, x) \in Y \times X : x \in F(y)\},$$

is a closed set in $Y \times X$.

DEFINITION 1.3 [15]. Let $f: Y \to X$ be a vector-valued map and $C \subset X$ a closed and convex cone. f is said to be C-quasiconvex if, for any $x \in X$, the set

$$M = \{y \in Y : f(y) \in x - C\}$$

is a convex subset in Y.

LEMMA 1.1 [23]. *Let C be a proper, closed and convex cone in locally convex Hausdorff topological vector space X and* $\operatorname{int} C \neq \varnothing$. *Its dual cone is defined by*

$$C^* = \{\phi \in X^* : \langle \phi, y \rangle \geqslant 0, \forall y \in C\},$$

where X^* is the dual space of X. Then

$$\begin{aligned} x \in C &\Longleftrightarrow \langle \phi, x \rangle \geqslant 0, \quad \forall \phi \in C^*; \\ x \in int\, C &\Longleftrightarrow \langle \phi, x \rangle > 0, \quad \forall \phi \in C^* \backslash \{0\}. \end{aligned}$$

The outline of the paper is as follows. In Section 2, a nonlinear scalarization function is introduced and its properties are discussed. In particular, lower and upper semi-continuity of the nonlinear scalarization function are established. In Section 3, the existence of solutions for vector quasi-equilibrium problems is obtained and an application is given to vector variational inequalities.

2. A Nonlinear Scalarization Function

Let X be a locally convex Hausdorff topological vector space. Let $C : X \to 2^X$ be a set-valued map and for any $x \in X$, $C(x)$ a proper, closed and convex cone with $\operatorname{int} C(x) \neq \varnothing$. Let $e : X \to X$ be a vector-valued map and for any $x \in X$, $e(x) \in \operatorname{int} C(x)$. Let X^* be the dual space of X, equipped with weakly star topology. Let $C^* : X \to 2^{X^*}$ be defined by, for any $x \in X$,

$$C^*(x) = \{\phi \in X^* : \langle \phi, y \rangle \geqslant 0, \forall y \in C(x)\}.$$

For any given $x \in X$, since $e(x) \in \operatorname{int} C(x)$, the set

$$B^*(x) = \{\phi \in C^*(x) : \langle \phi, e(x) \rangle = 1\}$$

is a weakly star compact base of the cone $C^*(x)$ (see [23]).

DEFINITION 2.1. The nonlinear scalarization function $\xi : X \times X \to R$ is defined by

$$\xi(x, z) = \inf\{\lambda \in R : z \in \lambda e(x) - C(x)\}, \quad \forall (x, z) \in X \times X.$$

REMARK 2.1. (i) Let S be a proper closed convex cone in X with $\operatorname{int} S \neq \varnothing$, and let $e \in \operatorname{int} S$. Recall the definition of the Gerstewitz function [10],

$$\xi_e(z) = \inf\{t \in R : z \in te - S\}, \quad z \in X.$$

If, for any $x \in X$, $C(x) = S$ and $e(x) = e$ in the Definition 2.1, the $\xi(x, z)$ reduces to $\xi_e(z)$.

(ii) Let $k^0 \in \operatorname{int} \bigcap_{x \in X} C(x) \neq \varnothing$. The scalariation function in [8] is defined as

$$\xi_{k^0}(x, z) = \inf\{t \in R : z \in tk^0 - C(x)\}.$$

We note that if, for any $x \in X$, $e(x) = k^0$, the function $\xi(x, z)$ reduces to $\xi_{k^0}(x, z)$. In the new definition, the assumption $\operatorname{int} \bigcap_{x \in X} C(x) \neq \varnothing$ is removed.

LEMMA 2.1. [8] *For each* $x \in X$,

$$X = \bigcup \{\lambda e(x) - \operatorname{int} C(x) : \lambda \in R^+ \backslash \{0\}\}.$$

LEMMA 2.2. *For* $\lambda \in R$ *and* $x \in X$, *set* $S_\lambda(x) = \lambda e(x) - C(x)$.

(i) *If* $z \in S_\lambda(x)$ *holds for some* $\lambda \in R$, *and* $x \in X$, *then*

$$z \in \mu e(x) - \operatorname{int} C(x), \quad \text{for each } \mu > \lambda;$$

moreover,

$$z \in \mu e(x) - C(x), \quad \text{for each } \mu > \lambda.$$

(ii) *For each* $x; z \in X$, *there exists a real number* $\lambda \in R$ *such that* $z \notin S_\lambda(x)$.

(iii) *Let* $z \in X$. *If* $z \notin S_\lambda(x)$ *for some* $\lambda \in R$, *and* $x \in X$, *then*

$$z \notin S_\mu(x), \quad \text{for each } \mu < \lambda.$$

Proof. The proof of this lemma is similar to that of Lemma 2.2 in [8].

PROPOSITION 2.1. *The function* $\xi : X \times X \to R$ *is well defined and*

$$\xi(x, z) = \min\{\lambda \in R : z \in \lambda e(x) - C(x)\}.$$

Proof. Using the key lemma, Lemma 2.2, the proof of this proposition is similar to that of Proposition 2.1 in [8].

PROPOSITION 2.2. *For any* $(x, z) \in X \times X$,

$$\xi(x, z) = \max_{\phi \in B^*(x)} \frac{\langle \phi, z \rangle}{\langle \phi, e(x) \rangle}.$$

Proof. We show firstly,

$$\xi(x,z)=\sup_{\phi\in C^*(x)\setminus\{0\}}\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}.$$

Since $\xi(x,z)=\min\{\lambda\in R: z\in\lambda e(x)-C(x)\}, z\in\xi(x,z)e(x)-C(x)$, equivalently,

$$\xi(x,z)e(x)-z\in C(x).$$

For any $\phi\in C^*(x)\setminus\{0\}\subset C^*(x)$, we have $\langle\phi,\xi(x,z)e(x)-z)\rangle\geqslant 0$, equivalently,

$$\xi(x,z)\langle\phi,e(x)\rangle-\langle\phi,z\rangle\geqslant 0.$$

Because $e(x)\in\operatorname{int}C(x)$ and $\phi\in C^*(x)\setminus\{0\}$, by Lemma 1.1, we have $\langle\phi,e(x)\rangle>0$. So $\xi(x,z)\geqslant\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}$. That is to say,

$$\xi(x,z)\geqslant\sup_{\phi\in C^*(x)\setminus\{0\}}\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}.$$

On the other hand, let

$$\lambda_0=\sup_{\phi\in C^*(x)\setminus\{0\}}\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}.$$

So, for any $\phi\in C^*\setminus\{0\}$, $\lambda_0\geqslant\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}$. Since $\langle\phi,e(x)\rangle>0$, $\langle\phi,\lambda_0 e(x)-z\rangle\geqslant 0$. By Lemma 1.1, $\lambda_0 e(x)-z\in C(x)$, i.e. $z\in\lambda_0 e(x)-C(x)$. From the definition of ξ, $\lambda_0\geqslant\xi(x,z)=\min\{\lambda\in R: z\in\lambda e(x)-C(x)\}$, i.e.

$$\xi(x,z)\leqslant\sup_{\phi\in C^*(x)\setminus\{0\}}\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}.$$

So we have

$$\xi(x,z)=\sup_{\phi\in C^*(x)\setminus\{0\}}\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}.$$

Since $B^*(x)$ is the base of $C^*(x)$, for any $x\in X,\phi\in C^*(x)\setminus\{0\}$, there is $\lambda>0$, and $\varphi\in B^*(x)$ such that $\phi=\lambda\varphi$. So for any $x\in X$,

$$\frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}=\frac{\langle\lambda\varphi,z\rangle}{\langle\lambda\varphi,e(x)\rangle}=\frac{\langle\varphi,z\rangle}{\langle\varphi,e(x)\rangle}.$$

So we have

$$\sup_{\phi\in C^*(x)\setminus\{0\}} \frac{\langle\phi, z\rangle}{\langle\phi, e(x)\rangle} = \sup_{\phi\in B^*(x)} \frac{\langle\phi, z\rangle}{\langle\phi, e(x)\rangle}.$$

i.e.

$$\xi(x,z) = \sup_{\phi\in B^*(x)} \frac{\langle\phi, z\rangle}{\langle\phi, e(x)\rangle}.$$

Since $B^*(x)$ is weakly star compact, $\xi(x,z) = \max_{\phi\in B^*(x)} \frac{\langle\phi,z\rangle}{\langle\phi,e(x)\rangle}$.

PROPOSITION 2.3. *For each $r\in R$ and $x,z\in X$, the following statements are satisfied.*

(i) $\xi(x,z) < r \Longleftrightarrow z\in \mathrm{re}(x) - \mathrm{int}\, C(x)$.
(ii) $\xi(x,z) \leqslant r \Longleftrightarrow z\in \mathrm{re}(x) - C(x)$.
(iii) $\xi(x,z) \geqslant r \Longleftrightarrow z\notin \mathrm{re}(x) - \mathrm{int}\, C(x)$.
(iv) $\xi(x,z) > r \Longleftrightarrow z\notin \mathrm{re}(x) - C(x)$.
(v) $\xi(x,z) = r \Longleftrightarrow z\in \mathrm{re}(x) - \partial C(x)$, *where $\partial C(x)$ is the topological boundary of $C(x)$.*

Proof. We only prove (i). The proofs for other assertions are similar and omitted. Suppose $\xi(x,z) < r$, i.e.,

$$\begin{aligned}
\xi(x,z) < r &\Longleftrightarrow \max_{\phi\in B^*(x)} \frac{\langle\phi, z\rangle}{\langle\phi, e(x)\rangle} < r\\
&\Longleftrightarrow \langle\phi, z\rangle < r\langle\phi, e(x)\rangle, \quad \forall\phi\in B^*(x)\\
&\Longleftrightarrow \langle\phi, re(x) - z\rangle > 0, \quad \forall\phi\in B^*(x)\\
&\Longleftrightarrow \langle\phi, re(x) - z\rangle > 0, \quad \forall\phi\in C^*(x)\setminus\{0\}\\
&\Longleftrightarrow re(x) - z\in \mathrm{int}\, C(x),\\
&\Longleftrightarrow z\in re(x) - \mathrm{int}\, C(x).
\end{aligned}$$

PROPOSITION 2.4. *Let X be a locally convex Hausdorff topological vector space, and for any given $x\in X$,*

(i) $\xi(x,\cdot)$ *is positively homogenous;*
(ii) $\xi(x,\cdot)$ *is strictly monotone, that is, if $z_1\in z_2+\mathrm{int}\, C(x)$, then*

$$\xi(x,z_2) < \xi(x,z_1).$$

Proof. (i) Let $\mu > 0$. For $z \in X$, we have

$$\begin{aligned}\xi(x, \mu z) &= \max_{\phi \in B^*(x)} \frac{\langle \phi, \mu z\rangle}{\langle \phi, e(x)\rangle} \\ &= \mu \max_{\phi \in B^*(x)} \frac{\langle \phi, z\rangle}{\langle \phi, e(x)\rangle} \\ &= \mu \xi(x, z).\end{aligned}$$

(ii) Let $z_1 \in z_2 + \operatorname{int} C(x)$. Set $r = \xi(x, z_1)$. By Proposition 2.3(v), we have

$$z_2 \in z_1 - \operatorname{int} C(x) \subset re(x) - C(x) - \operatorname{int} C(x) \subset re(x) - \operatorname{int} C(x).$$

By Proposition 2.3(i), we have

$$\xi(x, z_2) < r = \xi(x, z_1).$$

PROPOSITION 2.5. *For any fixed $x \in X$, and any $z_1, z_2 \in X$,*

(i) $\xi(x, z_1 + z_2) \leqslant \xi(x, z_1) + \xi(x, z_2)$;
(ii) $\xi(x, z_1 - z_2) \geqslant \xi(x, z_1) - \xi(x, z_2)$.

Proof.

(i)

$$\begin{aligned}\xi(x, z_1 + z_2) &= \max_{\phi \in B^*(x)} \frac{\langle \phi, z_1 + z_2\rangle}{\langle \phi, e(x)\rangle} \\ &\leqslant \max_{\phi \in B^*(x)} \frac{\langle \phi, z_1\rangle}{\langle \phi, e(x)\rangle} + \max_{\phi \in B^*(x)} \frac{\langle \phi, z_2\rangle}{\langle \phi, e(x)\rangle} \\ &= \xi(x, z_1) + \xi(x, z_2).\end{aligned}$$

(ii) It follows from (i) that

$$\xi(x, z_1) = \xi(x, z_1 - z_2 + z_2) \leqslant \xi(x, z_1 - z_2) + \xi(x, z_2).$$

Then, $\xi(x, z_1) - \xi(x, z_2) \leqslant \xi(x, z_1 - z_2)$, which implies that (ii) holds.

THEOREM 2.1. *Let X be a locally convex Hausdorff topological vector space, and let $C: X \to 2^X$ be a set-valued map such that, for each $x \in X$, $C(x)$ is a proper, closed, convex cone in X with* int $C(x) \neq \varnothing$. *And let $e: X \to X$ be the continuous selection of the set-valued map* $\operatorname{int} C(\cdot)$, *i.e. e is continuous and $e(x) \in \operatorname{int} C(x)$, for all $x \in X$. Define a set-valued map $W: X \to 2^X$ by $W(x) = X \setminus int C(x)$, for $x \in X$. We have*

(i) *If W is upper semi-continuous, then $\xi(\cdot,\cdot)$ is upper semi-continuous on $X \times X$;*
(ii) *If C is upper semi-continuous, then $\xi(\cdot,\cdot)$ is lower semi-continuous on $X \times X$.*

Proof. (i) In order to show $\xi(\cdot,\cdot)$ is upper semi-continuous, we must check, for any $\lambda \in R$, the set

$$A := \{(x,z) \in X \times X : \xi(x,z) \geqslant r\}$$

is closed. Let $(x_\alpha, z_\alpha) \in A$ and $(x_\alpha, z_\alpha) \to (x_0, z_0)$. We have $\xi(x_\alpha, z_\alpha) \geqslant r$, it is to say, by Proposition 2.3(iii),

$$z_\alpha \notin re(x_\alpha) - \operatorname{int} C(x_\alpha).$$

Namely, $\mathrm{re}(x_\alpha) - z_\alpha \in X \backslash \operatorname{int} C(x_\alpha) = W(x_\alpha)$. Since $e(\cdot)$ is continuous on X, $(\mathrm{re}(x_\alpha) - z_\alpha, x_\alpha) \to (\mathrm{re}(x_0) - z_0, x_0)$. Since W is upper semi-continuous with closed valued, by Proposition 7 (pp. 110) in [1], W is closed. So $\mathrm{re}(x_0) - z_0 \in W(x_0)$. Namely, $z_0 \notin \mathrm{re}(x_0) - \operatorname{int} C(x_0)$. By Proposition 2.3(iii), it is equivalent to $\xi(x_0, z_0) \geqslant r$. So, A is closed, i.e., $\xi(\cdot,\cdot)$ is upper semi-continuous on $X \times X$.

(ii) In order to show $\xi(\cdot,\cdot)$ is lower semi-continuous, we must check, for any $\lambda \in R$, the set

$$B := \{(x,z) \in X \times X : \xi(x,z) \leqslant r\}$$

is closed. Let $(x_\alpha, z_\alpha) \in B$ and $(x_\alpha, z_\alpha) \to (x_0, z_0)$. We have $\xi(x_\alpha, z_\alpha) \leqslant r$, it is to say, by Proposition 2.3(ii),

$$z_\alpha \in \mathrm{re}(x_\alpha) - C(x_\alpha).$$

Since $e(\cdot)$ is continuous on $X, (\mathrm{re}(x_\alpha) - z_\alpha, x_\alpha) \to (\mathrm{re}(x_0) - z_0, x_0)$. Since $C(\cdot)$ is upper semi-continuous with closed valued, by Proposition 7 (pp. 110) in [1], C is closed. So $\mathrm{re}(x_0) - z_0 \in C(x_0)$. Namely, $z_0 \in \mathrm{re}(x_0) - C(x_0)$. By Proposition 2.3 (ii), it is equivalent to $\xi(x_0, z_0) \leqslant r$. So, B is closed, i.e., $\xi(\cdot,\cdot)$ is lower semi-continuous on $X \times X$.

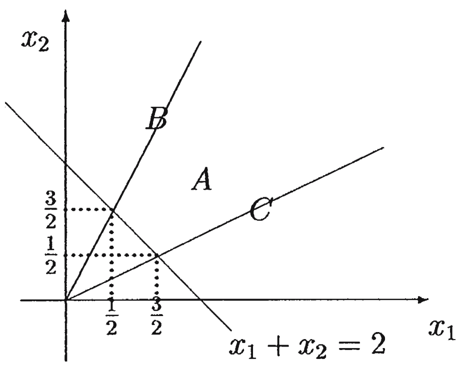
REMARK 2.2. (i) If X is a paracompact space, and $\mathrm{int}C^{-1}(y) = \{x \in X, y \in \operatorname{int} C(x)\}$ is an open set and for each $x \in X$, $\operatorname{int} C(x) \neq \varnothing$ and $C(x)$ is convex, by the Browder continuous select theorem, $\operatorname{int} C(\cdot)$ has a continuous select $e(\cdot)$.

(ii) If $e^0 \in \operatorname{int} \bigcap_{x \in X} C(x)$, we could let for any $x \in X, e(x) = e^0$. The function e is also continuous.

The following examples are to show that if $C(W$, respectively) is not upper semi-continuous, then $\xi(\cdot,\cdot)$ is not lower semi-continuous (upper semicontinuous, respectively) under the conditions that all the other conditions are satisfied.

EXAMPLE 2.1. Let $X=R^2$, the 2-dimensional Eulidean space. Let

$$A=\mathrm{Cone}\left(\left\{(x_1,x_2)\in R^2 : x_1+x_2=2,\ \frac{1}{2}\leqslant x_1\leqslant\frac{3}{2}\right\}\right),$$
$$B=\mathrm{Cone}\left(\left\{(x_1,x_2)\in R^2 : x_1+x_2=2,\ 0\leqslant x_1\leqslant\frac{3}{2}\right\}\right),$$
$$C=\mathrm{Cone}\left(\left\{(x_1,x_2)\in R^2 : x_1+x_2=2,\ \frac{1}{2}\leqslant x_1\leqslant 2\right\}\right).$$



The set-valued map $C: X\to 2^X$ is defined by

$$C((x_1,x_2))=\begin{cases} A, & \text{if } x_1=0;\\ B, & \text{if } x_1>0;\\ C, & \text{if } x_1<0.\end{cases}$$

$$W((x_1,x_2))=\begin{cases} X\backslash\mathrm{int}\ A, & \text{if } x_1=0;\\ X\backslash\mathrm{int}\ B, & \text{if } x_1>0;\\ X\backslash\mathrm{int}\ C, & \text{if } x_1<0.\end{cases}$$

Let $e=(1,1)$ and for any $x=(x_1,x_2)\in X$, $e(x)=e$.

Note that for any $x\in X$, $\mathrm{int}\,C(x)\neq\emptyset$ and $e\in\mathrm{int}\,C(x)$. We still note that $W(\cdot)$ is upper semi-continuous, so $\xi(\cdot,\cdot)$ is upper semi-continuous on

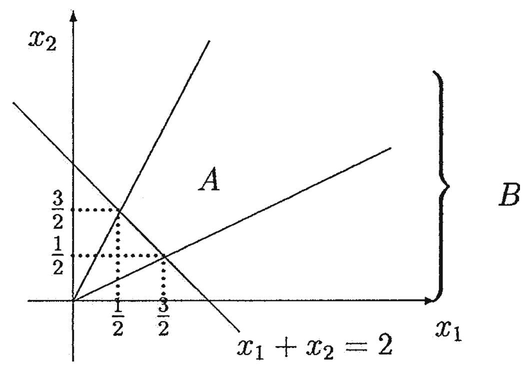
$X \times X$. But $C(\cdot)$ is not upper semi-continuous. Note that the level set of the function ξ at 0,

$$\begin{aligned} L(\xi,0) &= \{((x_1,x_2),(z_1,z_2)) \in R^2 \times R^2 : \xi((x_1,x_2),(z_1,z_2)) \leqslant 0\} \\ &= (\{(x_1,x_2) \in R^2 : x_1 = 0\} \times (-A)) \\ &\quad \cup (\{(x_1,x_2) \in R^2 : x_1 > 0\} \times (-B)) \\ &\quad \cup (\{(x_1,x_2) \in R^2 : x_1 < 0\} \times (-C)), \end{aligned}$$

is not a closed set. That is to say, $\xi(\cdot,\cdot)$ is not lower semi-continuous.

EXAMPLE 2.2. Let $X = R^2$, the 2-dimensional Eulidean space. Let

$$A = \mathrm{Cone}\left(\left\{(x_1,x_2) \in R^2 : x_1 + x_2 = 2, \frac{1}{2} \leqslant x_1 \leqslant \frac{3}{2}\right\}\right),$$
$$B = \mathrm{Cone}\left(\left\{(x_1,x_2) \in R^2 : x_1 + x_2 = 2, 0 \leqslant x_1 \leqslant 2\right\}\right),$$



The set-valued map $C : X \to 2^X$ is defined by

$$C((x_1,x_2)) = \begin{cases} B, & \text{if } x_1 = 0; \\ A, & \text{if } x_1 \neq 0. \end{cases}$$

$$W((x_1,x_2)) = \begin{cases} X\backslash \mathrm{int}\ B, & \text{if } x_1 = 0; \\ X\backslash \mathrm{int}\ A, & \text{if } x_1 \neq 0. \end{cases}$$

Let $e = (1,1)$ and for any $x = (x_1,x_2) \in X$, $e(x) = e$.

Note that for any $x \in X$, $\mathrm{int}\, C(x) \neq \emptyset$ and $e \in \mathrm{int}\, C(x)$. We still note that $C(\cdot)$ is upper semi-continuous, so $\xi(\cdot,\cdot)$ is lower semi-continuous on $X \times X$.

But $W(\cdot)$ is not upper semi-continuous. Note that the strict level set of the function ξ at 0,

$$\begin{aligned} L_s(\xi,0) &= \{((x_1,x_2),(z_1,z_2)) \in R^2 \times R^2 : \xi((x_1,x_2),(z_1,z_2)) < 0\} \\ &= (\{(x_1,x_2) \in R^2 : x_1 = 0\} \times (-\mathrm{int} B)) \\ &\quad \cup (\{(x_1,x_2) \in R^2 : x_1 \neq 0\} \times (-\mathrm{int} A)) \end{aligned}$$

is not a open set. That is to say, $\xi(\cdot,\cdot)$ is not upper semi-continuous.

3. Generalized Quasi-equilibrium Problem

In this section, we apply the nonlinear scalarization function to obtain the existence of solutions for vector quasi-equilibrium problems.

Let E, Z and X be Hausdorff topological vector spaces. Let $C: X \to 2^X$ be a set-valued map such that for every $x \in X, C(x)$ is a proper, closed and convex cone with a nonempty interior $\mathrm{int}\, C(x)$, i.e., for each $x \in X, (X, C(x))$ is an ordered space. Let $Y \subset E$ and $D \subset Z$ be nonempty sets. Let $Q: Y \to 2^Y$ and $V: Y \to 2^D$ be set-valued maps. Let $g: E \to X$ be a vector-valued map. Let $f: Y \times D \times Y \to X$ be a vector-valued map.

The following fixed point theorem plays an important tool in the establishment of the existence of generalized quasi-equilibrium problems.

THEOREM 3.1 (Fan-Glicksber-Kakutani) [1]. *Let Y be a nonempty compact subset of a locally convex Hausdorff vector topological space E. If $F: Y \to 2^Y$ is upper semi-continuous and for any $y \in Y$, $F(y)$ is a nonempty, convex and closed subset, then there exists a $\bar{y} \in Y$ such that $\bar{y} \in F(\bar{y})$.*

We consider the following generalized quasi-equilibrium problem(GQEP)

$$\begin{aligned} &\text{Find } \bar{y} \in Q(\bar{y}) \text{ and } \bar{z} \in V(\bar{y}) \text{ such that} \\ &f(\bar{y},\bar{z},\bar{y}) - f(\bar{y},\bar{z},y) \notin \mathrm{int}\, C(g(\bar{y})), \quad \forall y \in Q(\bar{y}). \end{aligned}$$

REMARK 3.1. Obviously, the problem (GQEP) in this paper is a generalization of the vector equilibrium problems considered in [3].

LEMMA 3.1. *If, for each $y \in Y, z \in Z$, the mapping $f(y,z,\cdot): v \mapsto f(y,z,v)$ is $C(g(y))$-quasi-convex, then the function $v \to \xi(g(y), f(y,z,v))$ is R_+-quasiconvex.*

Proof. For $t_0 \in R$, set

$$\mathrm{Lev}(t_0) = \{v \in Y : \xi(g(y), f(y,z,v)) \leqslant t_0\}.$$

It suffices to show that $\mathrm{Lev}(t_0)$ is a convex subset in E. Indeed, we suppose that $v_1, v_2 \in \mathrm{Lev}(t_0)$ and $\lambda \in [0, 1]$. Then,

$$\xi(g(y), f(y, z, v)) \leqslant t_0, \quad i = 1, 2.$$

This means that

$$f(y, z, v_i) \in t_0 e(g(y)) - C(g(y)), \quad i = 1, 2.$$

Let

$$M = \{v \in Y : f(y, z, v) \in t_0 e(g(y)) - C(g(y))\}.$$

Then $v_1, v_2 \in M$. By the $C(g(y))$-quasiconvexity of f, we have

$$\lambda v_1 + (1 - \lambda)v_2 \in M.$$

Therefore

$$f(y, z, \lambda v_1 + (1 - \lambda)v_2) \in t_0 e(g(y)) - C(g(y)),$$

By Proposition 2.3(ii), we have

$$\lambda v_1 + (1 - \lambda)v_2 \in \mathrm{Lev}(t_0).$$

and hence $v \to \xi(g(y), f(y, z, v))$ is R_+-quasiconvex.

Next, by using the nonlinear scalarization function we obtain an existence theorem of (GQEP) without any assumption of monotonicity.

THEOREM 3.2. *Let E, Z and X be locally convex Hausdorff topological vector spaces. Let $C : X \to 2^X$ be a set-valued map such that for every $x \in X, C(x)$ is a proper, closed and convex cone with a nonempty interior* $\mathrm{int}\, C(x)$. *Assume that* $\mathrm{int}\, C(\cdot)$ *has continuous select $e(\cdot)$. Define a set-valued map $W : X \to 2^X$ by $W(x) = X \backslash \mathrm{int}\, C(x)$, for $x \in X$. Let $Y \subset E$ and $D \subset Z$ be nonempty compact convex sets. Let $Q : Y \to 2^Y$ and $V : Y \to 2^D$ be set-valued maps. Let $g : E \to X$ be a vector-valued map. Let $f : Y \times D \times Y \to X$ be a vector-valued map. Suppose all the following conditions are satisfied.*

(i) *Both W and C are upper semi-continuous on X;*
(ii) *f and g are continuous on $Y \times D \times Y$ and E, respectively;*
(iii) *For each $y \in Y$ and $z \in D$ the mapping $v \to f(y, z, v)$ is $C(g(y))$-quasiconvex;*
(iv) *V is upper semi-continuous on Y;*
(v) *For each $y \in Y$, $V(y)$($Q(y)$, respectively) is a nonempty closed and convex subset of D(Y, respectively) and*
(vi) *Q is continuous on E;*

Then, there exist $\bar{y} \in Q(\bar{y})$ *and* $\bar{z} \in V(\bar{y})$ *such that*

$$f(\bar{y}, \bar{z}, \bar{y}) - f(\bar{y}, \bar{z}, y) \notin \operatorname{int} C(g(\bar{y})), \quad \forall y \in Q(\bar{y}).$$

Proof. Define a function:

$$\xi(g(y), f(y, z, v)) = \inf\{t \in R : f(y, z, v) \in te(g(y)) - C(g(y))\},$$

and a set-valued map $\Delta : Y \times D \to 2^Y$,

$$\Delta(y, z) = \left\{u \in Q(y) : \xi(g(y), f(y, z, u)) = \min_{v \in Q(y)} \xi(g(y), f(y, z, v))\right\}.$$

We shall show that

(a) Δ is upper semi-continuous on $Y \times D$;
(b) Δ is closed and convex-valued map.

For those, we observe that

$$\Delta(y, z) = \left\{u \in Q(y) : -\xi(g(y), f(y, z, u)) = \max_{v \in Q(y)} \xi(g(y), f(y, z, v))\right\}.$$

By Theorem 2.1 and the continuity of g, f, the function ξ is continuous on $Y \times D$. So, the assumptions of Proposition 23 of [1] hold and hence Δ is upper semi-continuous on $Y \times D$. Then, (a) holds.

To show (b), let $u_1, u_2 \in \Delta(y, z)$ and $\lambda \in (0, 1)$. Define

$$r_0 = \min_{v \in Q(y)} \xi(g(y), f(y, z, v)).$$

Then,

$$\xi(g(y), f(y, z, u_i)) = r_0, \quad i = 1, 2.$$

Since $Q(y)$ is convex, $\lambda u_1 + (1 - \lambda) u_2 \in Q(y)$. By the assumption (iii) and Lemma 3.1, the function $v \to \xi(g(y), f(y, z, v))$ is R_+-quasiconvex. Then, the set

$$M = \{(v \in Y : \xi(g(y), f(y, z, v) \leqslant r_0\}$$

is convex. Since $u_1, u_2 \in M$, $\lambda u_1 + (1 - \lambda) u_2 \in M$. Then

$$\xi(g(y), f(y, z, \lambda u_1 + (1 - \lambda) u_2) \leqslant r_0.$$

By the definition of r_0, we have

$$\xi(g(y), f(y, z, \lambda u_1 + (1 - \lambda) u_2) = r_0.$$

This implies that $\lambda u_1 + (1-\lambda)u_2 \in \Delta(y,z)$.

Now we define a set-valued map W: $Y \times D \to 2^{Y\times D}$,

$$W(y,z) = \Delta(y,z) \times V(y), \quad \forall (y,z) \in Y \times D.$$

Observe that W is convex valued, closed valued and upper semi-continuous. By Fan-Glicksberg-Kakutani theorem, there exists $(\bar{y},\bar{z}) \in W(\bar{y},\bar{z})$. Hence,

$$\bar{y} \in Q(\bar{y}), \quad \xi(g(\bar{y}), f(\bar{y},\bar{z},\bar{y}) = \min_{v \in Q(\bar{y})} \xi(g(\bar{y}), f(\bar{y},\bar{z},v))$$

and $\bar{z} \in V(\bar{y})$. Thus, we have

$$\xi(g(\bar{y}), f(\bar{y},\bar{z},v)) - \xi(g(\bar{y}), f(\bar{y},\bar{z},\bar{y})) \geqslant 0, \quad \forall v \in Q(\bar{y}).$$

By the Proposition 2.5, we have

$$\xi(g(\bar{y}), f(\bar{y},\bar{z},v) - f(\bar{y},\bar{z},\bar{y})) \geqslant 0, \quad \forall v \in Q(\bar{y}).$$

By the Proposition 2.3(iii),

$$f(\bar{y},\bar{z},\bar{y}) = -f(\bar{y},\bar{z},v) \notin \operatorname{int} C(g(\bar{y})), \quad \forall v \in Q(\bar{y}).$$

REMARK 3.2. Theorem 3.2 could be considered as the vectorial generalization of several generalized quasi-equilibrium problems or generalized quai-variational inequality problems, for example: Theorem 1 in [24], Theorem 2 in [14].

As an application of Theorem 3.2, consider the following vector variational inequality (VVI) [11, 20, 21]:

Find $\bar{y} \in Y, \quad s.t. \ \langle F(\bar{y}), y - \bar{y}\rangle \notin -\operatorname{int} C, \quad \forall y \in Y,$

where E and X is a locally convex Hausdorff topological vector space, C is a closed and convex cone in X, $F: E \to L(Y,X)$ is continuous and $Y \subset E$ is a nonempty, compact and convex set.

Let

$$\begin{aligned} f(y,z,v) &= \langle F(y), v\rangle \\ g: Y &\to X, \quad \text{any continuous mapping} \\ C(x) &= C, \forall x \\ Q(y) &= Y \\ V(y) &= V, \quad \text{a nonempty, closed and convex set} \\ D &= Z. \end{aligned}$$

Then, (GQEP) is reduced to (VVI). It is easy to check that the assumptions in Theorem 3.2 are satisfied:

(i) $C(x)$ is a constant cone, hence both W and C are upper semi-continuous;
(ii) F is continuous in y and linear in v, hence f is continuous in (y, z, v) and g is continuous on Y;
(iii) $v \to f(y, z, v) = \langle F(y), v \rangle$ is linear, so it is C-quasiconvex;
(iv) $V(y)$ is a constant mapping, hence upper semi-continuous;
(v) $Q(x) = Y \subset E$ is a nonempty, compact and convex set;
(vi) $Q(y)$ is a constant mapping, hence continuous. Thus, (VVI) has solution.

Note that this is an infinite dimensional space generalization of the result obtained in [9].

4. Conclusions

In this paper, we introduced a nonlinear scalarization function and established its lower semi-continuous and upper semi-continuous properties. We applied it to the study of vector quasi-equilibrium problems. The existence of a solution for vector quasi-equilibrium problems was obtained without any monotonicity condition.

Acknowledgment

The work in this paper was supported by a grant from the Research Grants Council of Hong Kong Special Administration Region, China (Poly U 5141/O1E).

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